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SLOWLY ROTATING RELATIVISTIC STARS. VI. STABILITY OF THE QUASI-RADIAL MODES

JAMES B. HARTLE*
University of California, Santa Barbara

KIP S. THORNE†
California Institute of Technology, Pasadena

AND

S. M. CHITRE‡
Tata Institute of Fundamental Research, Bombay, India, and
Institute for Space Studies, New York, New York
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ABSTRACT

Equations are given for calculating the effects of a slow and rigid rotation on the frequency of the radial modes of oscillation of a relativistic star. The rotation is treated to second order in the angular velocity, but no other approximations are made.

I. INTRODUCTION

Rotating, relativistic stars are today accepted as the underlying explanation of the pulsar phenomenon. Rotating relativistic objects may also be at the hearts of quasars (see, e.g., Morrison 1969) and some galactic X-ray sources (Margon *et al.* 1971). For a quantitative explanation of the observed phenomena it is not enough to be able to calculate the equilibrium configurations of relativistic stars. One must also understand how these stars pulsate.

Pulsations may help explain some of the observed periodic phenomena of pulsars such as the subpulse structure discovered by Drake and Craft (1968). Pulsations may play an important role in the evolution of a star to its final equilibrium form by the emission of gravitational radiation from modes which do not have a high degree of symmetry. Even when only the equilibrium configurations are of final interest, the study of their pulsations must still be pursued to distinguish the stable configurations from the unstable ones.

The radial pulsations of nonrotating stars have been analyzed by Chandrasekhar (1964), Bardeen (1965), Bardeen, Thorne, and Meltzer (1966), Meltzer and Thorne (1966), Faulkner and Gribben (1968), Cohen, Lapidus, and Cameron (1969) among others. Nonradial oscillations of nonrotating stars and their accompanying gravitational radiation have been studied by Thorne and Campolattaro (1967), Price and Thorne (1969), and Thorne (1969).

In this paper we begin a dynamical analysis of the quasi-radial modes of slowly and rigidly rotating, relativistic stars. The equilibrium theory for these stars was given in Papers I and II (Hartle 1967; Hartle and Thorne 1968). The quasi-radial modes are those modes which would be radial if the star were not rotating. They are the crucial modes for determining the stability of the rotating star and also the modes which store energy for the longest time against dissipation by gravitational radiation.

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‡ NAS-NRC Senior Research Associate.

If only the stability of these modes were at issue, a full dynamical pulsation analysis would not be necessary. A much simpler static stability criterion has already been given in Paper III of this series (Hartle and Thorne 1969) and has been applied to stellar models in Paper V (Munn and Hartle 1972). That analysis, however, involves only the calculation of equilibrium configurations and does not determine the eigenfrequencies, mode shapes, rate of radiation damping, or any other dynamical properties of a rotating pulsating star.

A dynamical analysis of the quasi-radial pulsations of a slowly rotating relativistic star is begun in this paper with a derivation of a prescription for calculating the rotation-induced change in pulsation frequency. In later papers of this series we expect to derive expressions for the rate of gravitational radiation from these modes and to apply our results to particular stellar models.

While simple in principle, a dynamical pulsation analysis of a slowly rotating star is algebraically complex. We therefore begin in § II with a general and abstract discussion of the principles of the method to be used. This is followed in § III by the definitions of the many quantities needed to describe the pulsating star and its gravitational field. An explicit expression for the change in frequency of a radial mode due to rotation is obtained in terms of these definitions in § IV. Finally, in § V we recover the Newtonian analysis (Chandrasekhar and Lebovitz 1968) of the pulsations of slowly rotating fluid masses.

II. FREQUENCY OF A QUASI-RADIAL MODE

The small oscillations of a relativistic star are conveniently described by the displacement $\zeta\boldsymbol{\xi}(\boldsymbol{x}, t)$ of a fluid element from its equilibrium position $\boldsymbol{x}$. Here, ζ is a formal expansion parameter proportional to the amplitude of vibration.[1] Associated with the oscillation there will be small changes in the pressure, density, and metric. The Eulerian changes in these quantities will be denoted[2] by $\zeta\delta\mathcal{P}$, $\zeta\delta\mathcal{E}$, and $\zeta\delta\boldsymbol{g}$, respectively. These quantities may be found by solving Einstein's equations accurate to first order in the amplitude of oscillation:

$$\zeta\delta\boldsymbol{E} \equiv \zeta\delta[\boldsymbol{R} - \tfrac{1}{2}\boldsymbol{g}R - 8\pi\boldsymbol{T}] = 0\,. \tag{2.1}$$

These equations will be linear differential equations for the unknown quantities $\boldsymbol{\xi}$, $\delta\mathcal{P}$, $\delta\mathcal{E}$, and $\delta\boldsymbol{g}$. One must supply, as input information for these equations, (i) the functions $\mathcal{E}$, $\mathcal{P}$, and $\boldsymbol{g}$ which describe the nonpulsating equilibrium configuration, (ii) the equation of state

$$\mathcal{E} = \mathcal{E}(\mathcal{P})\,, \tag{2.2}$$

and (iii) the adiabatic index Γ governing the pulsation

$$\Gamma = \frac{\mathcal{E} + \mathcal{P}}{\mathcal{P}}\left(\frac{\partial\mathcal{P}}{\partial\mathcal{E}}\right)_{\text{constant entropy}}\,. \tag{2.3}$$

Equation (2.1) describes small oscillations about equilibrium. Since time does not occur explicitly in this equation (stationary unperturbed configuration!), and since the equation is linear, one can find solutions which vary harmonically in time:

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \boldsymbol{\xi}(\boldsymbol{x})e^{i\sigma t}\,, \qquad \delta\boldsymbol{g}(\boldsymbol{x}, t) = \delta\boldsymbol{g}(\boldsymbol{x})e^{i\sigma t}\,, \qquad \text{etc.} \tag{2.4}$$

[1] Thus, terms multiplying ζ are linear in the displacement, terms multiplying ζ^2 are quadratic, etc. We set the value of ζ to 1 to permit dropping ζ from any expression where it is not needed as "bookkeeper."

[2] As far as possible we follow the notation of Paper I. Our conventions follow those of Landau and Lifshitz (1962) with $c = G = 1$ except that Greek indices range over space and time coordinates while Latin indices run only over the space variables. Boldface letters denote vectors and tensors; e.g., the displacement vector is $\boldsymbol{\xi} = \xi_\mu \boldsymbol{dx}^\mu$ and the metric is $\boldsymbol{g} = g_{\mu\nu}\boldsymbol{dx}^\mu \otimes \boldsymbol{dx}^\nu$.

In general, these oscillating solutions will contain both incoming and outgoing gravitational waves, as measured by the behavior of the metric at large distances from the star. However, for certain discrete complex values of the angular frequency, $\sigma = \sigma_R + i\sigma_I$, there will be solutions with only outgoing gravitational waves ("physically acceptable solutions"). If $\sigma_I < 0$, the solution is an unstable mode of oscillation. If $\sigma_I > 0$, it is a stable oscillation with a half-life of $2/\sigma_I$ for damping of vibration energy by the forces of gravitational radiation reaction.

For radial oscillations of a nonrotating star, no emission of gravitational waves is possible. Thus either $\sigma^2 \geq 0$ and the oscillations are stable, or $\sigma^2 < 0$ and they are unstable. If the star is now given a small angular velocity Ω, the number of modes of oscillation will not change, but the frequency of each mode will:

$$\sigma = \sigma(\Omega) . \tag{2.5}$$

Our aim here is to calculate the frequency of the quasi-radial modes as a function of Ω for small angular velocities. In this way we will determine how the stability of a star is affected by a slow rotation.

For slow rotations the change in the frequency can be studied by a perturbation analysis. We expand all quantities and equations in powers of a formal expansion parameter ϵ with value unity. Terms linear in ϵ are proportional to the angular velocity of rotation Ω, terms quadratic in ϵ are proportional to Ω^2, etc. For example, we write

$$\begin{aligned} \boldsymbol{\xi}(x) &= \boldsymbol{\xi}^{(0)}(x) + \epsilon^2 \boldsymbol{\xi}^{(2)}(x) + \ldots , \\ \sigma &= \sigma^{(0)} + \epsilon^2 \sigma^{(2)} + \ldots , \\ \delta \boldsymbol{g} &= \delta^{(0)} \boldsymbol{g} + \epsilon \delta^{(1)} \boldsymbol{g} + \epsilon^2 \delta^{(2)} \boldsymbol{g} + \ldots . \end{aligned} \tag{2.6}$$

Here, we have used the fact that a reversal of the direction of rotation cannot change the shape of the mode or its frequency, but will change some components of the metric (for example, those which describe the local rate of rotation of inertial frames). By expanding the vibrational Einstein equations (2.1) in powers of ϵ, one obtains sets of equations which, together with boundary conditions, determine the functions and frequencies $\boldsymbol{\xi}^{(0)}$, $\delta^{(0)}\boldsymbol{g}$, $\sigma^{(0)}$ (determined by $\epsilon^0\zeta$ part of eq. [2.1]), $\delta^{(1)}\boldsymbol{g}$ (determined by ζ part), and $\boldsymbol{\xi}^{(2)}$, $\delta^{(2)}\boldsymbol{g}$, $\sigma^{(2)}$ (determined by $\epsilon^2\zeta$ part).

The perturbation of a radial mode caused by rotation is axially symmetric. The radiation field is therefore also axially symmetric. It follows that no angular momentum is lost through gravitational radiation (see, e.g., Price and Thorne 1969 for a discussion of this point). The star's constant angular momentum determines *uniquely* the large-distance behavior of those terms in the metric which are of odd order in ϵ (see box 19.1 of Misner, Thorne, and Wheeler 1972). Consequently, there can be no gravitational radiation to order ϵ.

The lowest-order metric coefficient which contains radiation is thus of order ϵ^2. The *rate* of radiation is then proportional to ϵ^4, since the rate is a bilinear functional of the gravitational-wave fields. In this way we conclude that, calculated to order ϵ^2 there is no gravitational damping and the square of the frequency remains real.

An expression for the change in frequency of pulsation to order ϵ^2 can be found by considering the functional form of the vibrational Einstein equations (eq. [2.1]). The equations can be decomposed as

$$\zeta \delta^{(0)} \boldsymbol{E} = (\text{linear in order } \zeta) , \tag{2.6a}$$

$$\zeta \delta^{(1)} \boldsymbol{E} = (\text{linear in order } \epsilon\zeta) + (\text{products of order } \epsilon \text{ with order } \zeta) , \tag{2.6b}$$

$$\begin{aligned} \epsilon^2 \zeta \delta^{(2)} \boldsymbol{E} = {} & (\text{linear in order } \epsilon^2\zeta) \\ & + (\text{products of order } \epsilon^2 \text{ with order } \zeta \text{ and of order } \epsilon\zeta \text{ with order } \epsilon) . \end{aligned} \tag{2.6c}$$

The important point for the following is that all of the linear terms in the above equations have the same functional form. That is, for example, the terms in the equation $\epsilon^2\zeta\delta^{(2)}\boldsymbol{E}$ which involve quantities of order $\epsilon^2\zeta$ have the same functional form in terms of those quantities as the equation $\zeta\delta^{(0)}\boldsymbol{E}$ has in the quantities of order ζ. The first term in each of equations (2.6) involves the same linear operation.

The equations which determine the change in squared frequency $\sigma^{(2)}$ are those of order $\epsilon^2\zeta$, equation (2.6c). They are linear in the unknowns of that order: $\boldsymbol{\xi}^{(2)}$, $\delta^{(2)}\boldsymbol{g}$, $\delta^{(2)}\mathcal{P}$, etc. They involve driving terms, which are products of quantities of lesser order—quantities which are known already, from previous studies of pulsating nonrotating stars, of rotating nonpulsating stars, and from the equations of order $\epsilon\zeta$. Some of equations (2.6c) can be used to eliminate all the unknowns of order $\epsilon^2\zeta$ except $\boldsymbol{\xi}^{(2)}$. The single equation for $\boldsymbol{\xi}^{(2)}$ then takes the form

$$\mathcal{L}(\boldsymbol{\xi}^{(2)}) + (\sigma^2)^{(2)}\boldsymbol{N} = \boldsymbol{D}\,, \tag{2.7}$$

where $\mathcal{L}$ is a linear operator and $\boldsymbol{N}$ and $\boldsymbol{D}$ are driving terms involving sums of products of quantities of lesser order than $\epsilon^2\zeta$.

As argued above, the linear operator $\mathcal{L}$ will be the same as that which governs the radial oscillations of a nonrotating star:

$$\mathcal{L}(\boldsymbol{\xi}^{(0)}) = 0\,. \tag{2.8}$$

This equation has been studied by Chandrasekhar (1964). He finds that $\mathcal{L}$ is Hermitian in the sense that, for any two radial displacements $\boldsymbol{\lambda}$ and $\boldsymbol{n}$ and an appropriate weight function w,

$$\int d^3xw[\boldsymbol{\lambda}\cdot\mathcal{L}(\boldsymbol{n})] = \int d^3xw[\mathcal{L}(\boldsymbol{\lambda})\cdot\boldsymbol{n}]\,. \tag{2.9}$$

An expression for $(\sigma^2)^{(2)}$ can then be obtained by multiplying equation (2.7) by $\boldsymbol{\xi}^{(0)}$, integrating, and using equations (2.8) and (2.9):

$$(\sigma^2)^{(2)} = \frac{\int d^3xw\boldsymbol{\xi}^{(0)}\cdot\boldsymbol{D}}{\int d^3xw\boldsymbol{\xi}^{(0)}\cdot\boldsymbol{N}}\,. \tag{2.10}$$

Thus, it is not necessary to *solve* the equations of order $\epsilon^2\zeta$ in order to calculate how the frequency of a radial mode is changed by rotation. One has only to find the *form* of these equations, to determine thereby $\boldsymbol{N}$ and $\boldsymbol{D}$, and to perform the quadrature over quantities of lower order given in equation (2.10). We will make this procedure explicit in the following sections.

The above analysis can be simplified by choosing a polar axis along the rotation axis and expanding all quantities in spherical harmonics. For example,

$$\epsilon^2\zeta\delta\mathcal{P}^{(2)} = \epsilon^2\zeta[\delta\mathcal{P}_0^{(2)}(r) + \delta\mathcal{P}_2^{(2)}(r)P_2(\cos\theta) + \ldots]\,. \tag{2.11}$$

The expansions of $\delta\boldsymbol{g}$, $\boldsymbol{\xi}$ will involve the appropriate tensor and vector spherical harmonics of various orders l. Each of the linear terms in equation (2.6) involves a linear differential operator constructed from the spherically symmetric metric of the nonrotating, nonpulsating star. This operator cannot, therefore, couple different l-values. By contrast, the quadratic driving terms *will* couple different spherical harmonics according to the familiar law for addition of angular momenta in quantum mechanics. In the case of equation (2.6b) the input to the driving term of order ζ contains only $l = 0$ terms, since it describes the radial pulsation of a spherical star. The input of order ϵ contains only $l = 1$ terms—essentially because these terms give the angular velocity of the local inertial frames (a vector quantity; see Paper I). Thus, in equation (2.6b) the ζ terms ($l = 0$) and ϵ terms ($l = 1$) can combine quadratically only to form $l = 1$

driving terms. Conclusion: the quantities driven by those terms—the $\epsilon\zeta$ part of $\delta^{(1)}\boldsymbol{g}$ and $\boldsymbol{\xi}^{(1)}$—must be $l = 1$. In the same way, the $\epsilon^2\zeta$ quadratic driving terms of equation (2.6c) are either products of terms of orders ζ and ϵ^2, or of orders $\epsilon\zeta$ and ϵ. In the first case we are combining $l = 0$ and $l = 0$ or 2 (see Paper I), and in the second the combination is $l = 1$ and $l = 1$. Thus, the driving term in equation (2.6c) and hence the expansion of $\delta^{(2)}\boldsymbol{g}$ can contain at most $l = 0$, 1, or 2.

Return now to equation (2.10) from which the change in squared frequency is calculated. Since $\boldsymbol{\xi}^{(0)}$ is radial and a function of r only, only the $l = 0$ radial part of $\boldsymbol{D}$ or $\boldsymbol{N}$ will remain following the angular integration. Denoting the radial part of $\boldsymbol{\xi}^{(0)}$ by $\xi^{(0)}$ and the $l = 0$ radial parts of $\boldsymbol{N}$ and $\boldsymbol{D}$ by N_0 and D_0, we have

$$(\sigma^2)^{(2)} = \left(\int_0^R \xi^{(0)} D_0 w r^2 dr\right) \Big/ \left(\int_0^R \xi^{(0)} N_0 w r^2 dr\right) . \tag{2.12}$$

Notice that, to find the functions D_0 and N_0 needed for calculating the change in frequency, one need not write down *all* of the equations of order $\epsilon^2\zeta$. Only the $l = 0$, $\epsilon^2\zeta$ equations need be examined.

III. METRIC, STRESS-ENERGY, AND RICCI TENSORS

An enormous number of parameters are required in our description of a pulsating, rotating, relativistic star. In this section we shall introduce them and show how they enter into the metric tensor, the stress-energy tensor, and the Ricci tensor. For a summary see tables 1 and 2. As argued in § II, only the $l = 0$ parts of the metric are relevant to a determination of the change in the frequency. For convenience and for future reference, however, we give here the complete decomposition of all the relevant quantities.

a) Coordinate System and Form of the Metric

We begin our detailed analysis by constructing a coordinate system in which the metric takes on a particularly simple form. Our coordinate system will reduce, in the

TABLE 1

SPHERICAL HARMONICS USED IN THIS PAPER[a]

Type	l, Parity	Values
Scalar..........	0, +	$Y^0{}_0{}^* = 1$
	2, +	$Y^2{}_0{}^* = P_2 = \frac{1}{2}(3\cos^2\theta - 1)$
Vector.	1, +	$\Phi^1{}_0{}^*{}_\theta = 0\,, \quad \Phi^1{}_0{}^*{}_\varphi = -\sin^2\theta$
	2, +	$\Psi^2{}_0{}^*{}_\theta = -3\cos\theta\sin\theta\,, \quad \Psi^2{}_0{}^*{}_\varphi = 0$
Tensor.........	0, +	$\Phi^0{}_0{}^*{}_{\theta\theta} = 1\,, \quad \Phi^0{}_0{}^*{}_{\varphi\varphi} = \sin^2\theta\,, \quad \Phi^0{}_0{}^*{}_{\theta\varphi} = 0$
	2, + (first type)	$\Phi^2{}_0{}^*{}_{\theta\theta} = P_2 = \frac{1}{2}(3\cos^2\theta - 1)$ $\Phi^2{}_0{}^*{}_{\varphi\varphi} = \sin^2\theta P_2 = \frac{1}{2}(3\cos^2\theta - 1)\sin^2\theta$ $\Phi^2{}_0{}^*{}_{\theta\varphi} = 0$
	2, + (second type)	$\Psi^2{}_0{}^*{}_{\theta\theta} = 6\sin^2\theta - 3$ $\Psi^2{}_0{}^*{}_{\varphi\varphi} = -3\sin^2\theta\cos^2\theta$ $\Psi^2{}_0{}^*{}_{\theta\varphi} = 0$

[a] These are the spherical harmonics of Regge and Wheeler (1957; see also Appendix A of Thorne and Campolattaro 1967), except that they have been renormalized. The asterisk indicates renormalization: $\Psi^{l*}{}_{0AB} = [4\pi/(2l + 1)]^{1/2}\Psi^l{}_{0AB}$, etc. The superscript (0, 1, or 2) is the value of the harmonic index l; the subscript 0 is the value of the projection index m; the subscripts θ and φ are tensor indices. Included here are all nonvanishing scalar, vector, and tensor harmonics of order $l = 0, 1, 2$; $m = 0$; and parity $\pi = +$.

TABLE 2

PARAMETERS DESCRIBING A ROTATING, PULSATING STAR

LEVEL IN EXPANSION; FUNCTIONAL DEPENDENCE	ORIGIN OF PARAMETER: Metric (eq. [3.7])	Fluid Motion (eqs. [3.8], [3.9])	Density, Pressure (eqs. [3.11], [3.12], [3.15])
Nonrotating:			
$O(1)$, function of r	ν, λ	...	E, P, γ, Γ
Rotating, nonpulsating:			
$O(\epsilon)$, function of r	$\bar{\omega}$	$\Omega = \text{const.}$	...
$O(\epsilon^2)$, function of r	m_0, m_2, v_2, h_0, h_2	...	p_0^*, p_2^*
Pulsating, nonrotating:			
$O(\zeta)$, function of r, t	η, μ	U	...
Rotating, pulsating:			
$O(\epsilon\zeta)$, function of r, t	j_1	F	...
$O(\epsilon^2\zeta)$, function of r, t	H_0, H_2, K_2, N_0, N_2, Q_2	W_0, W_2, V	...

Other parameters:	
Spacetime coordinates	t, r, θ, φ (eq. [3.7])
Expansion parameters	ϵ, ζ (§ II)
Pulsation angular frequency	$\sigma^{(0)}, \sigma^{(2)}$ (eq. [4.5])
Legendre polynomial (see also table 1)	$P_2 = \frac{1}{2}(3\cos^2\theta - 1)$

case of no rotation, to that of Chandrasekhar (1964) for a radially pulsating star. In the case of no pulsation it will reduce to that of Paper I for a slowly rotating star.

We introduce coordinates t, r, θ, φ which satisfy two conditions: (i) at order $\epsilon^0\zeta^0$ they become the familiar spherical polar coordinates for a nonrotating, equilibrium stellar model:

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad \text{to} \quad O(\epsilon^0\zeta^0)\ ; \tag{3.0}$$

and (ii) to all orders in ϵ and ζ, φ is a cyclic azimuthal angle about the axis of symmetry. The metric is then independent of φ, and any transformation of the form

$$t \rightarrow f_1(t, r, \theta)\ , \quad r \rightarrow f_2(t, r, \theta)\ , \quad \theta \rightarrow f_3(t, r, \theta)\ , \quad \varphi \rightarrow \varphi + f_4(t, r, \theta) \tag{3.1}$$

will preserve this independence. The four arbitrary functions in this change of coordinates can be used to put the four constraints

$$g_{r\theta} = g_{t\theta} = g_{\theta\varphi} = 0\ , \quad g_{\varphi\varphi} = g_{\theta\theta}\sin^2\theta$$

on the metric coefficients, thereby bringing the line element into the form

$$ds^2 = -S_1dt^2 + S_2dr^2 + S_3drdt + r^2S_4[d\theta^2 + \sin^2\theta(d\varphi - L_1dt - L_2dr)^2]\ . \tag{3.2}$$

If an expansion of the metric in powers of the angular velocity is now made, the functions S_i will contain only terms that are even in ϵ, and L_i only terms that are odd. This follows from symmetry of the metric under the simultaneous reversal $\varphi \rightarrow -\varphi$, $\Omega \rightarrow -\Omega$.

It was shown in Paper I that for a rotating, nonpulsating star the metric terms even in ϵ contain only spherical harmonics of order $l = 0$ and 2, and of parity $\pi = +$; while terms odd in ϵ contain only order $l = 1$ and parity $\pi = +$. In the case of radial pulsation but not rotation (Chandrasekhar 1964), all metric terms have $l = 0$ and $\pi = +$. When rotation and pulsation are both present [terms of $O(1)$, $O(\zeta)$, $O(\epsilon)$, $O(\epsilon^2)$, $O(\epsilon\zeta)$, $O(\epsilon^2\zeta)$ present], the usual rules for coupling angular momentum outlined in § II dictate that terms of $O(\epsilon\zeta)$, like terms of $O(\epsilon)$, have $l = 1$ and $\pi = +$; while terms of $O(\epsilon^2\zeta)$, like terms of $O(\epsilon^2)$, have $l = 0$ or 2 and $\pi = +$, or possibly $l = 1$, $\pi = +$.

Combining the conclusions of the above two paragraphs, we conclude that the functions S_i are even in ϵ [$O(1)$, $O(\zeta)$, $O(\epsilon^2)$, $O(\epsilon^2\zeta)$] and have $l = 0$, 1, or 2, $\pi = +$; while the functions L_i are odd in ϵ [$O(\epsilon)$, $O(\epsilon\zeta)$] and have $l = 1$, $\pi = +$.

The spherical harmonics that enter into S_i and L_i are not all scalar harmonics. Of the metric components, g_{tt}, g_{rr}, and g_{tr} transform as scalars under rotations; g_{tA} and g_{rA} (with $A = \theta, \varphi$) transform as vectors; and g_{AB} (with A and B running over θ, φ) transform as a tensor. The relevant vector and tensor spherical harmonics have been introduced by Regge and Wheeler (1957) (see also Appendix A of Thorne and Campolattaro 1967). They are all summarized for the cases of interest in table 1. Axial symmetry dictates that all the harmonics for our problem have projection quantum number $m = 0$.

Since g_{tt}, g_{tr}, and g_{rr} are scalars under rotation, the corresponding functions S_1, S_2, and S_3 (eq. [3.2]) have terms independent of θ, and terms proportional to $P_2(\cos\theta)$. The coordinate conditions $g_{\theta\varphi} = 0$ and $g_{\varphi\varphi} = g_{\theta\theta}\sin^2\theta$ guarantee that g_{AB} involves ${\Phi^0}_0{}^*{}_{AB}$ and ${\Phi^2}_0{}^*{}_{AB}$, but not ${\Psi^2}_0{}^*{}_{AB}$—which in turn guarantees that S_4 has terms independent of θ, and terms proportional to $P_2(\cos\theta)$. The conditions $g_{t\theta} = g_{r\theta} = 0$ guarantee that g_{tA} and g_{rA} involve ${\Phi^1}_0{}^*{}_A$ but not ${\Psi^2}_0{}^*{}_A$—which in turn guarantees that L_1 and L_2 are independent of θ. In summary:

$$S_i(r, \theta, t) = S_{i0}(r, t) + S_{i2}(r, t)P_2(\cos\theta) + O(\epsilon^4)\,, \quad L_i(r, \theta, t) = L_{i1}(t, r) + O(\epsilon^3)\,. \tag{3.3}$$

The form (3.3) of the metric is still preserved by the transformations

$$t \to t + g_1(r, t)\,, \quad r \to r + g_2(r, t)\,, \quad \varphi \to \varphi + g_3(r, t)\,. \tag{3.4}$$

These transformations can be used to enforce the further conditions

$$S_{30} = 0\,, \quad S_{40} = 1\,, \quad L_{21} = 0\,. \tag{3.5}$$

The remaining terms in the metric (3.2) may now be expanded in powers of ϵ and ζ. The functions of order ζ^0 have already been defined in Papers I and II. Retaining those definitions, we define the remaining metric components as follows:

$$\begin{aligned}
g_{tt} &= -e^{\nu}(1 + \zeta\eta)[1 + 2\epsilon^2(h_0 + h_2P_2) + \epsilon^2\zeta(N_0 + N_2P_2)] + \epsilon^2r^2\sin^2\theta(\Omega - \bar{\omega})^2\,,\\
g_{rr} &= e^{\lambda}[1 + 2\epsilon^2e^{\lambda}(m_0 + m_2P_2)/r + 2\zeta e^{\lambda}\mu/r - \epsilon^2\zeta(H_0 + H_2P_2)]\\
g_{\theta\theta} &= g_{\varphi\varphi}/\sin^2\theta = r^2[1 + 2\epsilon^2(v_2 - h_2)P_2 - \epsilon^2\zeta K_2P_2]\,,\\
g_{tr} &= -\epsilon^2\zeta Q_2P_2\,, \qquad g_{t\varphi} = -\epsilon r^2\sin^2\theta(\Omega - \bar{\omega} + \zeta j_1)\,,\\
g_{t\theta} &= g_{r\theta} = g_{r\varphi} = g_{\theta\varphi} = 0\,.
\end{aligned} \tag{3.7}$$

The quantities η, N_0, N_2, Q_2, j_1, μ, H_0, H_2, and K_2 introduced here are all functions of r and t; while the quantities ν, λ, h_0, h_2, $\bar{\omega}$, m_0, m_2, and v_2, which are taken over from Papers I and II, are functions of r alone.

b) Fluid Motions

In a rotating but nonpulsating star the fluid moves in the φ-direction with angular velocity $\Omega = d\varphi/dt = u^{\varphi}/u^t$. When the star is set pulsating quasi-radially, its fluid gets displaced from its rotating equilibrium state by a certain amount ξ^r in the radial direction, and by smaller amounts (due to rotational flattening and Coriolis forces), ξ^θ and ξ^φ, in the tangential directions. The resultant four-velocity has the form

$$u^r \equiv \xi^r{}_{,t}u^t\,, \quad u^\theta \equiv \xi^\theta{}_{,t}u^t\,, \quad u^\varphi \equiv (\epsilon\Omega + \xi^\varphi{}_{,t})u^t\,, \quad g_{\alpha\beta}u^\alpha u^\beta = -1\,. \tag{3.8}$$

Here ξ^r, ξ^θ, and ξ^φ are functions of r, θ, and t, and $X_{,t}$ denotes $\partial X/\partial t$. Once again symmetry under the simultaneous reversal $\varphi \to -\varphi$, $\Omega \to -\Omega$ shows that ξ^r and ξ^θ contain only even terms in ϵ, while ξ^φ contains only odd. Examination of the spherical harmonics with parity $\pi = +$ (table 1) reveals that ξ^r, being a scalar under rotations, must have $l = 0$ and 2; while ξ^A (with $A = \theta$ and φ), being a vector, can have $l = 1$ and 2. More particularly, introducing new functions U, V, F, W_0, and W_2 of r and t, we can write

$$\xi^r = \zeta r^{-2}[e^{\nu/2}U + \epsilon^2 e^{-\lambda/2}(W_0 + W_2P_2)]\,, \quad \xi^\theta = \epsilon^2\zeta V\Psi^2{}_0{}^{*\theta} = -\epsilon^2\zeta 3V\cos\theta\sin\theta\,,$$
$$\xi^\varphi = \epsilon\zeta F\Phi^1{}_0{}^{*\varphi} = -\epsilon\zeta F\,. \tag{3.9}$$

Here we have used the two-sphere metric ($\gamma_{\theta\theta} = 1$, $\gamma_{\theta\varphi} = 0$, $\gamma_{\varphi\varphi} = \sin^2\theta$) to raise the indices on the vector spherical harmonics of table 1; and we have imposed the demand for radial pulsation at order ζ.

By combining the displacements (3.9) with the metric (3.7), in the manner dictated by equations (3.8), we obtain the following components for the fluid four-velocity:

$$\begin{aligned}
u^t &= e^{-\nu/2}\{1 - \tfrac{1}{2}\zeta\eta - \epsilon^2[(h_0 + h_2P_2) - \tfrac{1}{2}e^{-\nu}r^2\sin^2\theta\bar{\omega}^2] \\
&\quad + \epsilon^2\zeta[\tfrac{1}{2}\eta(h_0 + h_2P_2) - \tfrac{1}{2}(N_0 + N_2P_2) - e^{-\nu}r^2\sin^2\theta(\tfrac{3}{4}\eta\bar{\omega}^2 + j_1\Omega + F_{,t}(\bar{\omega}))]\}\,, \\
u^r &= \zeta r^{-2}U_{,t} + \epsilon^2\zeta\{[-r^{-2}(h_0 + h_2P_2) + \tfrac{1}{2}e^{-\nu}\sin^2\theta\bar{\omega}^2]U_{,t} \\
&\quad + r^{-2}e^{-(\lambda+\nu)/2}(W_{0,t} + W_{2,t}P_2)\}\,, \\
u^\theta &= -3\epsilon^2\zeta e^{-\nu/2}V_{,t}\cos\theta\sin\theta\,, \\
u^\varphi &= \epsilon e^{-\nu/2}\Omega - \epsilon\zeta e^{-\nu/2}[\tfrac{1}{2}\eta\Omega + F_{,t}]\,.
\end{aligned} \tag{3.10}$$

c) *Density and Pressure*

Let $\mathcal{E}$ and $\mathcal{P}$ be the total density of mass-energy and the pressure, as measured by an observer comoving with the fluid. Adopting the notation of Papers I and II for the density and pressure in a rotating, nonpulsating star, and letting $\delta\mathcal{E}$ and $\delta\mathcal{P}$ be the Eulerian changes due to pulsation, we have

$$\begin{aligned}
\mathcal{P} &= P + \epsilon^2(E + P)(p_0{}^* + p_2{}^*P_2) + \delta\mathcal{P}\,, \\
\mathcal{E} &= E + \epsilon^2[(E + P)^2/\gamma P](p_0{}^* + p_2{}^*P_2) + \delta\mathcal{P}\,.
\end{aligned} \tag{3.11}$$

Here $E = E(r)$ and $P = P(r)$ are density and pressure in the nonrotating, nonpulsating star; $p_0{}^* = p_0{}^*(r)$ and $p_2{}^* = p_2{}^*(r)$ are dimensionless rotational corrections, and

$$\gamma \equiv [(E + P)/P](dP/dE)_{\text{eq. state}} \tag{3.12}$$

is the "adiabatic index" associated with the equation of state (see Papers I and II).

The Eulerian changes due to pulsation, $\delta\mathcal{E}$ and $\delta\mathcal{P}$, can be calculated by the following procedure: (i) Calculate the fractional Lagrangian change in the number density of baryons, $\Delta\mathfrak{N}/\mathfrak{N}$, by imposing baryon-number conservation, $[\mathfrak{N}u^\alpha]_{;\alpha} \propto [\mathfrak{N}(-g)^{1/2}u^\alpha]_{,\alpha} = 0$. Note that this conservation law can be put in the more convenient form

$$\frac{\Delta\mathfrak{N}}{\mathfrak{N}} = -\frac{\delta[(-g)^{1/2}u^t] + [(-g)^{1/2}u^t\xi^j]_{,j}}{(-g)^{1/2}u^t}\,, \tag{3.13}$$

where δ denotes the Eulerian change due to pulsation—i.e., the part proportional to the pulsational expansion parameter, ζ. (ii) Next use the first law of thermodynamics for adiabatic pulsations,

$$\Delta\mathcal{E} = (\mathcal{E} + \mathcal{P})(\Delta\mathfrak{N}/\mathfrak{N}) \; ; \quad \Delta\mathcal{P} = \Gamma\mathcal{P}(\Delta\mathfrak{N}/\mathfrak{N}) \, , \tag{3.14}$$

to calculate the Lagrangian changes in $\mathcal{E}$ and $\mathcal{P}$. Here

$$\Gamma = (\mathfrak{N}/\mathcal{P})(d\mathcal{P}/d\mathfrak{N})_{\text{const. entropy}} \tag{3.15}$$

is the adiabatic index for pulsations—not to be confused with the formally defined index (3.12) for the equation of state. (iii) Finally, infer the Eulerian changes in $\mathcal{P}$ and $\mathcal{E}$ from the Lagrangian changes:

$$\delta\mathcal{P} = \Delta\mathcal{P} - \mathcal{P}_{,j}\xi^j \, , \quad \delta\mathcal{E} = \Delta\mathcal{E} - \mathcal{E}_{,j}\xi^j \, . \tag{3.16}$$

The calculation yields the following expressions for $\delta\mathcal{E}$ and $\delta\mathcal{P}$:

$$\begin{aligned} \delta\mathcal{E} = {} & \zeta[(E + P)\Upsilon_{\mathrm{NR}} - E' r^{-2} e^{\nu/2} U] \\ & + \epsilon^2\zeta\Big\{(E + P)\Upsilon_{\mathrm{R}} + (E + P)\left(\frac{E + P}{\gamma P} + 1\right)(p_0{}^* + p_2{}^* P_2)\Upsilon_{\mathrm{NR}} \\ & - \left[\frac{(E + P)^2}{\gamma P}(p_0{}^* + p_2{}^* P_2)\right]' r^{-2} e^{\nu/2} U - E' r^{-2} e^{-\lambda/2}(W_0 + W_2 P_2)\Big\} \, , \end{aligned} \tag{3.17}$$

$$\begin{aligned} \delta\mathcal{P} = {} & \zeta[\Gamma P \Upsilon_{\mathrm{NR}} - P' r^{-2} e^{\nu/2} U] + \epsilon^2\zeta\{\Gamma P \Upsilon_{\mathrm{R}} + \Gamma(E + P)(p_0{}^* + p_2{}^* P_2)\Upsilon_{\mathrm{NR}} \\ & - [(E + P)(p_0{}^* + p_2{}^* P_2)]' r^{-2} e^{\nu/2} U - P' r^{-2} e^{-\lambda/2}(W_0 + W_2 P_2)\} \, . \end{aligned} \tag{3.18}$$

Here Υ_{NR} is the value of $\Delta\mathfrak{N}/\mathfrak{N}$ for a pulsating but nonrotating star:

$$\Upsilon_{\mathrm{NR}} = -r^{-2} e^{-\lambda/2}[e^{(\lambda+\nu)/2} U]' - r^{-1} e^{\lambda}\mu \; ; \tag{3.19}$$

and Υ_R is the rotational correction to $\Delta\mathfrak{N}/\mathfrak{N}$ (i.e., terms of order $\epsilon^2\zeta$ in $\Delta\mathfrak{N}/\mathfrak{N}$):

$$\begin{aligned} \Upsilon_{\mathrm{R}} = {} & -r^{-2} e^{-\lambda/2}(W_0' + W_2' P_2) + 2 r^{-2} e^{2\lambda}(m_0 + m_2 P_2)\mu \\ & - r^{-2} e^{\nu/2}[r^{-1} e^{\lambda}(m_0 + m_2 P_2) + 2(v_2 - h_2) P_2 + \tfrac{1}{2} r^2 e^{-\nu}\bar{\omega}^2 \sin^2\theta]' U \\ & + 6 V P_2 + K_2 P_2 + \tfrac{1}{2}(H_0 + H_2 P_2) + r^2 e^{-\nu}(\tfrac{1}{2}\bar{\omega}^2\eta + \bar{\omega} j_1 + \bar{\omega} F_{,t}) \sin^2\theta \, . \end{aligned} \tag{3.20}$$

A prime denotes a radial derivative: $X' \equiv \partial X/\partial r$.

d) *Stress-Energy Tensor*

The stress-energy tensor for our rotating, pulsating star is given by the usual perfect-fluid expression

$$T^{\alpha\beta} = (\mathcal{E} + \mathcal{P}) u^\alpha u^\beta + g^{\alpha\beta}\mathcal{P} \, . \tag{3.21}$$

Here $g^{\alpha\beta}$ is the inverse of the covariant metric (eq. [3.7]); u^α is given by equations (3.10); and $\mathcal{E}$ and $\mathcal{P}$ are given by equations (3.11), (3.17), and (3.18). We do not give the components of $T^{\alpha\beta}$ explicitly here, because they would occupy about three pages; and the reader can calculate them fairly easily.

e) *Ricci Tensor*

Not so easy to calculate is the Ricci tensor for the metric (3.7). We would never contemplate calculating it by hand. However, it is a manageable task using the FORMAC capabilities of an IBM 360 computer. The computers at Caltech and the University of California at Santa Barbara have given us the covariant components for the Ricci tensor that are shown in table 3.

Notice that we have broken each component up into independent parts. Each part is characterized by a particular order ($\epsilon^a \zeta^b$, $a \leq 2$ and $b \leq 1$) in the perturbation expansion, and by a particular spherical-harmonic angular dependence.

IV. $l = 0$ AND $l = 1$ EQUATIONS OF MOTION

a) *Equations of* $O(1)$, $O(\epsilon)$, $O(\epsilon^2)$, *and* $O(\zeta)$

The Einstein field equations at $O(1)$ govern the structure of the nonrotating, nonpulsating star. After some manipulation they reduce to the familiar equations

$$e^{-\lambda} = 1 - 2M/r\,, \qquad M = \int_0^r 4\pi r^2 E dr\,, \tag{4.1a}$$

$$\frac{dP}{dr} = -\frac{(E+P)(M+4\pi r^3 P)}{r(r-2M)}\,, \tag{4.1b}$$

$$\frac{d\nu}{dr} = \frac{2(M+4\pi r^3 P)}{r(r-2M)}\,. \tag{4.1c}$$

We use these together with the equation of state, $P = P(E)$, to construct our unperturbed stellar model (cf. § II*c* of Paper II).

At $O(\epsilon)$ and $O(\epsilon^2)$ the field equations become the familiar equations for the structure of a slowly and rigidly rotating star, as derived and integrated in Papers I and II. These equations determine the functions $\bar{\omega}$, m_0, m_2, v_2, h_0, h_2, p_0^*, and p_2^* (cf. table 2).

At $O(\zeta)$ the field equations reduce to the equations for a radially pulsating, nonrotating star, which were first derived by Chandrasekhar (1964). We summarize those equations here:

The metric functions μ and η are determined uniquely in terms of the displacement function U by the "initial-value equations"

$$\mu = -4\pi(E+P)e^{\nu/2}U\,, \tag{4.2a}$$

$$\eta' = -8\pi e^{\lambda+\nu/2}[r^{-1}\Gamma P U' + r^{-2}(1+\tfrac{1}{2}r\nu')(E+P)U]\,. \tag{4.2b}$$

Equation (4.2b) is subject to the boundary condition

$$\eta = 0 \quad \text{outside the star}\,. \tag{4.2c}$$

The displacement function U evolves in accordance with the Chandrasekhar equation of motion

$$\begin{aligned} e^{(\nu+3\lambda)/2}r^{-2}(E+P)U_{,tt} &= [e^{(3\nu+\lambda)/2}r^{-2}\Gamma P U']' \\ &+ e^{(3\nu+\lambda)/2}\left[-\frac{4P'}{r^3} - 8\pi e^{\lambda}\frac{P(E+P)}{r^2} + \frac{P'^2}{r^2(E+P)}\right]U\,, \end{aligned} \tag{4.2d}$$

which is subject to the boundary conditions (cf. Bardeen, Thorne, and Meltzer 1966, eq. [7b])

$$U \propto r^3 \quad \text{near} \quad r = 0\,, \qquad \Gamma P U' = 0 \quad \text{at} \quad r = R \quad \text{(stellar surface)}\,. \tag{4.2e}$$

TABLE 3

NONVANISHING COVARIANT COMPONENTS OF THE RICCI TENSOR[a]

A. Terms of O(1) --- (Nonrotating Star)

$$R_{tt} = e^{\nu-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'}{r} + \frac{\nu'(\nu' - \lambda')}{4}\right], \qquad R_{rr} = -\frac{\nu''}{2} + \frac{\lambda'}{r} - \frac{\nu'(\nu' - \lambda')}{4}$$

$$R_{\theta\theta} = \frac{R_{\varphi\varphi}}{\sin^2\theta} = 1 - e^{-\lambda} - \frac{1}{2} r e^{-\lambda}(\nu' - \lambda')$$

B. Terms of O(ϵ), ℓ = 1 Part --- (Rotating, Nonpulsating Star)

$$\frac{R_{tA}}{\epsilon \Phi^{1*}_{0\,A}} = (\Omega - \bar{\omega}) + \frac{1}{2} e^{-\lambda} r^2\left[\bar{\omega}'' - \left(\frac{\nu' + \lambda'}{2} - \frac{4}{r}\right)\bar{\omega}' - \left(\frac{\nu' - \lambda'}{r} + \frac{2}{r^2}\right)(\Omega - \bar{\omega})\right]$$

C. Terms of O(ϵ^2), ℓ = 0 Part --- (Rotating, Nonpulsating Star)

$$\frac{R_{tt}}{\epsilon^2} = e^{\nu-\lambda}\left\{h_0'' + \left(\nu' - \frac{\lambda'}{2} + \frac{2}{r}\right)h_0' + \left[\nu'' + \frac{2\nu'}{r} + \frac{\nu'(\nu' - \lambda')}{2}\right]h_0\right\}$$
$$+ e^{\nu}\left[-\frac{\nu'}{2r} m_0' - \left(\frac{\nu''}{r} + \frac{3\nu'}{2r^2} + \frac{\nu'^2}{2r}\right)m_0\right] + \frac{2}{3} r^2 e^{-\lambda}(\Omega - \bar{\omega})\,\bar{\omega}'' - \frac{1}{3} r^2 e^{-\lambda}\,\bar{\omega}'^2$$
$$+ \frac{2}{3}\left[4re^{-\lambda} - r^2 e^{-\lambda}\left(\frac{\nu' + \lambda'}{2}\right)\right](\Omega - \bar{\omega})\,\bar{\omega}' + \frac{2}{3}\left[1 - e^{-\lambda} - r e^{-\lambda}\left(\frac{\nu' - \lambda'}{2}\right)\right](\Omega - \bar{\omega})^2$$

$$\frac{R_{rr}}{\epsilon^2} = -h_0'' + \left(\frac{\lambda'}{2} - \nu'\right)h_0' + e^{\lambda}\left(\frac{2}{r^2} + \frac{\nu'}{2r}\right)m_0' + e^{\lambda}\left(-\frac{2}{r^3} - \frac{\nu'}{2r^2} + \frac{2\lambda'}{r^2} + \frac{\nu'\lambda'}{2r}\right)m_0 + \frac{1}{3} e^{-\nu} r^2\,\bar{\omega}'^2$$

$$\frac{R_{AB}}{\epsilon^2 \Phi^{0*}_{0\,AB}} = m_0' + \left(\frac{1}{r} + \nu'\right)m_0 - r e^{-\lambda} h_0' - \frac{1}{6} r^4 e^{-\nu-\lambda}\,\bar{\omega}'^2$$

D. Terms of O(ϵ^2), ℓ = 2 Part --- (Rotating, Nonpulsating Star)

$$\frac{R_{tt}}{\epsilon^2 P_2(\cos\theta)} = e^{\nu-\lambda}\left\{h_2'' + \left(\nu' - \frac{\lambda'}{2} + \frac{2}{r}\right)h_2' + \left[\nu'' + \frac{2\nu'}{r} + \frac{\nu'(\nu' - \lambda')}{2} - \frac{6e^{\lambda}}{r^2}\right]h_2 + \nu'(v_2' - h_2')\right\}$$
$$+ e^{\nu}\left[-\frac{\nu'}{2r} m_2' - \left(\frac{\nu''}{r} + \frac{3\nu'}{2r^2} + \frac{\nu'^2}{2r}\right)m_2\right] - \frac{2}{3} r^2 e^{-\lambda}(\Omega - \bar{\omega})\,\bar{\omega}'' + \frac{1}{3} r^2 e^{-\lambda}\,\bar{\omega}'^2$$
$$- \frac{2}{3}\left[4re^{-\lambda} - r^2 e^{-\lambda}\left(\frac{\nu' + \lambda'}{2}\right)\right](\Omega - \bar{\omega})\,\bar{\omega}' - \frac{2}{3}\left[1 - e^{-\lambda} - r e^{-\lambda}\left(\frac{\nu' - \lambda'}{2}\right)\right](\Omega - \bar{\omega})^2$$

$$\frac{R_{rr}}{\epsilon^2 P_2(\cos\theta)} = h_2'' + \left(\frac{4}{r} - \nu' - \frac{\lambda'}{2}\right)h_2' - 2v_2'' + v_2'\left(-\frac{4}{r} + \lambda'\right)$$
$$+ e^{\lambda}\left(\frac{2}{r^2} + \frac{\nu'}{2r}\right)m_2' + e^{\lambda}\left(\frac{6e^{\lambda} - 2}{r^3} - \frac{\nu'}{2r^2} + \frac{2\lambda'}{r^2} + \frac{\nu'\lambda'}{2r}\right)m_2 - \frac{1}{3} r^2 e^{-\nu}\,\bar{\omega}'^2$$

$$\frac{R_{rA}}{\epsilon^2 \Psi^{2*}_{0\,A}} = -v_2' + \left(\frac{1}{r} - \frac{\nu'}{2}\right)h_2 + e^{\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{2r}\right)m_2$$

$$\frac{R_{AB}}{\epsilon^2 \Psi^{2*}_{0\,AB}} = -h_2 - \frac{e^{\lambda}}{r} m_2 + \frac{1}{6} r^4 e^{-\lambda-\nu}\,\bar{\omega}'^2$$

$$\frac{R_{AB}}{\epsilon^2 \Phi^{2*}_{0\,AB}} = e^{-\lambda}\left\{r^2(h_2'' - v_2'') + \left[r^2\left(\frac{\nu' - \lambda'}{2}\right) + 4r\right](h_2' - v_2') - rh_2' + \left[2 - 6e^{\lambda} + r(\nu' - \lambda')\right](h_2 - v_2)\right\}$$
$$+ m_2' + \left(\frac{1}{r} + \nu'\right)m_2 + \frac{2}{3} e^{-\nu-\lambda} r^4\,\bar{\omega}'^2$$

[a] The Ricci tensor is the sum of parts A through H. A prime denotes derivative with respect to r; a dot denotes derivative with respect to t. The indices A and B on R run over θ and φ.

E. Terms of $O(\zeta)$ --- (Pulsating, Nonrotating Star)

$$\frac{R_{tt}}{\zeta} = e^{\nu-\lambda}\left\{\frac{1}{2}\eta'' + \left[\frac{\nu'}{2} - \frac{\lambda'}{4} + \frac{1}{r}\right]\eta' + \left[\frac{\nu''}{2} + \frac{\nu'}{r} + \frac{\nu'(\nu'-\lambda')}{4}\right]\eta\right\}$$
$$+ e^{\nu}\left\{-\frac{1}{2}\frac{\nu'}{r}\mu' - \left[\frac{\nu''}{r} + \frac{3\nu'}{2r^2} + \frac{\nu'^2}{2r}\right]\mu\right\} - \frac{e^{\lambda}}{r}\ddot{\mu}$$

$$\frac{R_{rr}}{\zeta} = -\frac{1}{2}\eta'' + \left(\frac{\lambda'}{4} - \frac{\nu'}{2}\right)\eta' + e^{\lambda}\left[\left(\frac{2}{r^2} + \frac{\nu'}{2r}\right)\mu' + \left(-\frac{2}{r^3} - \frac{\nu' - 4\lambda'}{2r^2} + \frac{\nu'\lambda'}{2r}\right)\mu\right] + \frac{e^{2\lambda-\nu}}{r}\ddot{\mu}$$

$$\frac{R_{\theta\theta}}{\zeta} = \frac{R_{\varphi\varphi}}{\zeta\sin^2\theta} = -\frac{1}{2}r\,e^{-\lambda}\eta' + \mu' + \left(\frac{1}{r} + \nu'\right)\mu$$

$$\frac{R_{tr}}{\zeta} = 2r^{-2}e^{\lambda}\dot{\mu}$$

F. Terms of $O(\epsilon\,\zeta)$, $\ell = 1$ Part --- (Rotating, Pulsating Star)

$$\frac{R_{tA}}{\epsilon\zeta\Phi^{1\,*}_{0\,A}} = e^{-\lambda}\left\{-\frac{r^2}{2}j_1'' + \left[-2r + r^2\left(\frac{\nu'+\lambda'}{4}\right)\right]j_1' - \left[1 - e^{\lambda} + r\left(\frac{\nu'-\lambda'}{2}\right)\right]j_1\right\} - e^{-\lambda}\left[\frac{r^2}{4}\bar{\omega}' + \frac{r}{2}(\Omega - \bar{\omega})\right]\eta'$$
$$+ \left[-\frac{r}{2}\bar{\omega}' + (\Omega - \bar{\omega})\right]\mu' + \left[-r\,\bar{\omega}'' + \left(-\frac{7}{2} + \frac{r}{2}\nu'\right)\bar{\omega}' + \left(\frac{1}{r} + \nu'\right)(\Omega - \bar{\omega})\right]\mu$$

$$\frac{R_{rA}}{\epsilon\zeta\Phi^{1\,*}_{0\,A}} = -\frac{r^2}{2}e^{-\nu}\dot{j}_1' - \frac{r^2}{4}e^{-\nu}\bar{\omega}'\dot{\eta} - \frac{r}{2}e^{-\nu+\lambda}\bar{\omega}'\dot{\mu}$$

G. Terms of $O(\epsilon^2\zeta)$, $\ell = 0$ Part --- (Rotating, Pulsating Star)

$$\frac{R_{tt}}{\epsilon^2\zeta} = \frac{1}{2}\ddot{H}_0 + e^{\nu-\lambda}\left\{\frac{1}{2}N_0'' + \left(\frac{1}{r} + \frac{\nu'}{2} - \frac{\lambda'}{4}\right)N_0' + \frac{\nu'}{4}H_0' + \left[\frac{\nu''}{2} + \frac{\nu'}{r} + \frac{\nu'(\nu'-\lambda')}{4}\right](H_0 + N_0)\right\}$$
$$+ \frac{2}{3}e^{-\lambda}\left\{\left[-r^2\bar{\omega}' + \left(2r - r^2\frac{\nu'}{2}\right)(\Omega - \bar{\omega})\right]j_1' + \left[\left(-2r + r^2\frac{\nu'}{2}\right)\bar{\omega}' + \left(4 + 2e^{\lambda} - 3r\nu' + r^2\frac{\nu'^2}{2}\right)(\Omega - \bar{\omega})\right]j_1\right\}$$
$$+ e^{\nu-\lambda}\left(-\frac{e^{\lambda}}{r}m_0 + h_0\right)\eta'' + e^{\nu-\lambda}\left\{e^{\lambda}\left[-\frac{1}{2r}m_0' - \left(\frac{3}{2r^2} + \frac{\nu'}{r}\right)m_0\right] + h_0' + \left(\frac{2}{r} + \nu' - \frac{\lambda'}{2}\right)h_0\right.$$
$$\left. - \frac{1}{3}e^{-\nu}\left[r^2(\Omega - \bar{\omega})\bar{\omega}' + r(\Omega - \bar{\omega})^2\right]\right\}\eta'$$
$$+ e^{\nu-\lambda}\left\{e^{\lambda}\left[-\frac{\nu'}{2r}m_0' - \left(\frac{\nu''}{r} + \frac{3\nu'}{2r^2} + \frac{\nu'^2}{2r}\right)m_0\right] + h_0'' + \left(\frac{2}{r} + \nu' - \frac{\lambda'}{2}\right)h_0' + \left[\nu'' + \frac{2\nu'}{r} + \frac{\nu'(\nu'-\lambda')}{2}\right]h_0\right\}\eta$$
$$+ e^{\nu}\left\{2e^{\lambda}\frac{\nu'}{r^2}m_0 - \frac{1}{r}h_0' - \frac{\nu'}{r}h_0 + \frac{2}{3}e^{-\nu}\left[-r(\Omega - \bar{\omega})\bar{\omega}' + (\Omega - \bar{\omega})^2\right]\right\}\mu'$$
$$+ e^{\nu}\left\{e^{\lambda}\left[\frac{2\nu'}{r^2}m_0' + \left(\frac{4\nu''}{r^2} + \frac{4\nu'}{r^3} + \frac{2\nu'(\nu'+\lambda')}{r^2}\right)m_0\right] - \frac{2}{r}h_0'' - \left(\frac{3}{r^2} + \frac{2\nu'}{r}\right)h_0' - \left(\frac{2\nu''}{r} + \frac{3\nu'}{r^2} + \frac{\nu'^2}{r}\right)h_0\right.$$
$$\left. + \frac{2}{3}e^{-\nu}\left[-2r(\Omega - \bar{\omega})\bar{\omega}'' + (-7 + r\nu')(\Omega - \bar{\omega})\bar{\omega}' + r\bar{\omega}'^2 + \left(\frac{1}{r} + \nu'\right)(\Omega - \bar{\omega})^2\right]\right\}\mu + \frac{2}{r^2}e^{2\lambda}m_0\ddot{\mu}$$

$$\frac{R_{rr}}{\epsilon^2\zeta} = -\frac{1}{2}e^{-\nu+\lambda}\ddot{H}_0 - \left(\frac{1}{r} + \frac{\nu'}{4}\right)H_0' - \frac{1}{2}N_0'' - \left(\frac{\nu'}{2} - \frac{\lambda'}{4}\right)N_0' - \frac{2}{3}e^{-\nu}r^2(\Omega - \bar{\omega})j_1''$$
$$+ \frac{2}{3}e^{-\nu}\left\{r^2\bar{\omega}' + \left[-4r + r^2\left(\nu' + \frac{\lambda'}{2}\right)\right](\Omega - \bar{\omega})\right\}j_1' + \frac{2}{3}e^{-\nu}\left\{r^2\bar{\omega}'' + \left[4r - r^2\left(\nu' + \frac{\lambda'}{2}\right)\right]\bar{\omega}'\right.$$
$$\left. + \left[-2 + r^2\nu'' + r(2\nu' + \lambda') - r^2\frac{\nu'\lambda'}{2}\right](\Omega - \bar{\omega})\right\}j_1 + \left\{e^{\lambda}\left[\frac{1}{2r}m_0' + \left(-\frac{1}{2r^2} + \frac{\lambda'}{2r}\right)m_0\right] - h_0'\right\}\eta'$$
$$- \frac{1}{3}e^{-\nu}r^2\bar{\omega}'^2\eta + e^{\lambda}\left[-e^{\lambda}\left(\frac{4}{r^3} + \frac{\nu'}{r^2}\right)m_0 + \frac{1}{r}h_0'\right]\mu'$$
$$+ e^{\lambda}\left\{e^{\lambda}\left[-\left(\frac{4}{r^3} + \frac{\nu'}{r^2}\right)m_0' + \left(\frac{8}{r^4} + \frac{2\nu' - 8\lambda'}{r^3} - \frac{2\nu'\lambda'}{r^2}\right)m_0\right] + \left(-\frac{1}{r^2} + \frac{\lambda'}{r}\right)h_0'\right\}\mu - \frac{2}{r}e^{-\nu+2\lambda}h_0\ddot{\mu}$$

$$\frac{R_{tr}}{\epsilon^2\zeta} = -\frac{1}{r}\dot{H}_0 - \frac{1}{3}e^{-\nu}r^2(\Omega - \bar{\omega})\dot{j}_1' - \frac{1}{6}e^{-\nu}r^2(\Omega - \bar{\omega})\bar{\omega}'\dot{\eta} - \left[\frac{1}{3}e^{-\nu+\lambda}r(\Omega - \bar{\omega})\bar{\omega}' + \frac{4}{r^3}e^{2\lambda}m_0\right]\dot{\mu}$$

$$\frac{R_{AB}}{\epsilon^2\zeta\Phi^{0\,*}_{0\,AB}} = -\frac{1}{2}e^{-\lambda}\left\{r(N_0' + H_0') + [2 + r(\nu' - \lambda')]H_0\right\} + \frac{2}{3}e^{-\nu-\lambda}\left[\frac{1}{2}r^4\bar{\omega}' - r^3(\Omega - \bar{\omega})\right]j_1'$$
$$+ \frac{2}{3}e^{-\nu-\lambda}[r^3\bar{\omega}' - (2r^2 - r^3\nu')(\Omega - \bar{\omega})]j_1 + m_0\eta' + \frac{1}{6}e^{-\nu-\lambda}r^4\bar{\omega}'^2\eta$$
$$- \frac{4}{r}e^{\lambda}m_0\mu' + \left\{-\frac{4}{r}e^{\lambda}\left[m_0' + (\nu' + \lambda')m_0\right] + 2h_0' + \frac{1}{3}e^{-\nu}r^3\bar{\omega}'^2\right\}\mu$$

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H. Terms of $O(\epsilon^2\zeta)$, $\ell = 2$ Part --- (Rotating, Pulsating Star)

$$
\begin{aligned}
\frac{R_{tt}}{\epsilon^2\zeta P_2} = {} & \frac{1}{2}\ddot{H}_2 + \ddot{K}_2 + e^{\nu-\lambda}\bigg\{\frac{1}{2}N_2'' + \left(\frac{1}{r} + \frac{\nu'}{2} - \frac{\lambda'}{4}\right)N_2' + \frac{\nu'}{4}H_2' - \frac{\nu'}{2}K_2' + \left[\frac{\nu''}{2} + \frac{\nu'}{r} + \frac{\nu'(\nu'-\lambda')}{4}\right](H_2 + N_2) \\
& - \frac{3}{r^2}e^{\lambda}N_2\bigg\} \\
& - e^{-\lambda}\left[\dot{Q}_2' + \left(\frac{2}{r} - \frac{\lambda'}{2}\right)\dot{Q}_2\right] - \frac{2}{3}e^{-\lambda}\left[-r^2\bar{\omega}' + \left(2r - r^2\frac{\nu'}{2}\right)(\Omega - \bar{\omega})\right]j_1' \\
& - \frac{2}{3}e^{-\lambda}\left[\left(-2r + r^2\frac{\nu'}{2}\right)\bar{\omega}' + \left(4 - 4e^{\lambda} - 3r\nu' + r^2\frac{\nu'^2}{2}\right)(\Omega - \bar{\omega})\right]j_1 + e^{\nu-\lambda}\left(-\frac{e^{\lambda}}{r}m_2 + h_2\right)\eta'' \\
& + e^{\nu-\lambda}\bigg\{e^{\lambda}\left[-\frac{1}{2r}m_2' - \left(\frac{3}{2r^2} + \frac{\nu'}{r}\right)m_2\right] + \left(\frac{2}{r} + \nu' - \frac{\lambda'}{2}\right)h_2 + v_2' + \frac{1}{3}e^{-\nu}\left[r^2(\Omega - \bar{\omega})\bar{\omega}' + r(\Omega - \bar{\omega})^2\right]\bigg\}\eta' \\
& + e^{\nu-\lambda}\bigg\{e^{\lambda}\left[-\frac{\nu'}{2r}m_2' - \left(\frac{\nu''}{r} + \frac{3\nu'}{2r^2} + \frac{\nu'^2}{2r}\right)m_2\right] + h_2'' + \left(\frac{2}{r} - \frac{\lambda'}{2}\right)h_2' + \bigg[-\frac{6}{r^2}e^{\lambda} + \nu'' \\
& + \frac{2\nu'}{r} + \frac{\nu'(\nu'-\lambda')}{2}\bigg]h_2 + \nu' v_2'\bigg\}\eta \\
& + e^{\nu}\bigg\{2e^{\lambda}\frac{\nu'}{r^2}m_2 - \frac{1}{r}h_2' - \frac{\nu'}{r}h_2 - \frac{2}{3}e^{-\nu}\left[-r(\Omega - \bar{\omega})\bar{\omega}' + (\Omega - \bar{\omega})^2\right]\bigg\}\mu' \\
& + e^{\nu}\bigg\{e^{\lambda}\left[\frac{2\nu'}{r^2}m_2' + \left(\frac{4\nu''}{r^2} + \frac{4\nu'}{r^3} + \frac{2\nu'(\nu'+\lambda')}{r^2}\right)m_2\right] - \frac{2}{r}h_2'' - \frac{3}{r^2}h_2' - \left(\frac{2\nu''}{r} + \frac{3\nu'}{r^2} + \frac{\nu'^2}{r}\right)h_2 - \frac{2\nu'}{r}v_2' \\
& - \frac{2}{3}e^{-\nu}\left[-2r(\Omega - \bar{\omega})\bar{\omega}'' + (-7 + r\nu')(\Omega - \bar{\omega})\bar{\omega}' + r\bar{\omega}'^2 + \left(\frac{1}{r} + \nu'\right)(\Omega - \bar{\omega})^2\right]\bigg\}\mu + \frac{2}{r^2}e^{2\lambda}m_2\ddot{\mu}
\end{aligned}
$$

$$
\begin{aligned}
\frac{R_{rr}}{\epsilon^2\zeta P_2} = {} & -\frac{1}{2}e^{-\nu+\lambda}\ddot{H}_2 - \left(\frac{1}{r} + \frac{\nu'}{4}\right)H_2' - \frac{3}{r^2}e^{\lambda}H_2 - \frac{1}{2}N_2'' - \left(\frac{\nu'}{2} - \frac{\lambda'}{4}\right)N_2' + K_2'' + \left(\frac{2}{r} - \frac{\lambda'}{2}\right)K_2' + e^{-\nu}\left(\dot{Q}_2' - \frac{\lambda'}{2}\dot{Q}_2\right) \\
& + \frac{2}{3}r^2e^{-\nu}(\Omega - \bar{\omega})j_1'' - \frac{2}{3}e^{-\nu}\bigg\{r^2\bar{\omega}' + \left[-4r + r^2\left(\nu' + \frac{\lambda'}{2}\right)\right](\Omega - \bar{\omega})\bigg\}j_1' - \frac{2}{3}e^{-\nu}\bigg\{r^2\bar{\omega}'' + \left[4r - r^2\left(\nu' + \frac{\lambda'}{2}\right)\right]\bar{\omega}' \\
& + \left[-2 + r^2\nu'' + r(2\nu' + \lambda') - r^2\frac{\nu'\lambda'}{2}\right](\Omega - \bar{\omega})\bigg\}j_1 + \bigg\{e^{\lambda}\left[\frac{1}{2r}m_2' + \left(-\frac{1}{2r^2} + \frac{\lambda'}{2r}\right)m_2\right] - h_2'\bigg\}\eta' \\
& + \frac{1}{3}e^{-\nu}r^2\bar{\omega}'^2\eta + e^{\lambda}\left[-e^{\lambda}\left(\frac{4}{r^3} + \frac{\nu'}{r^2}\right)m_2 + \frac{2}{r}v_2' - \frac{1}{r}h_2'\right]\mu' \\
& + e^{\lambda}\bigg\{e^{\lambda}\left[-\left(\frac{4}{r^3} + \frac{\nu'}{r^2}\right)m_2' + \left(\frac{8}{r^4} + \frac{2\nu' - 8\lambda'}{r^3} - \frac{2\nu'\lambda'}{r^2}\right)m_2\right] + \left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)\left(h_2' - 2v_2'\right)\bigg\}\mu - \frac{2}{r}e^{-\nu+2\lambda}h_2\ddot{\mu}
\end{aligned}
$$

$$
\begin{aligned}
\frac{R_{tr}}{\epsilon^2\zeta P_2} = {} & e^{-\lambda}\left[-\frac{3}{r^2}e^{\lambda} + \frac{\nu''}{2} + \frac{\nu'}{r} + \frac{\nu'(\nu'-\lambda')}{4}\right]Q_2 + \dot{K}_2' + \left(\frac{1}{r} - \frac{\nu'}{2}\right)\dot{K}_2 - \frac{1}{r}\dot{H}_2 + \frac{r^2}{3}e^{-\nu}(\Omega - \bar{\omega})\dot{j}_1' \\
& + \frac{1}{6}r^2e^{-\nu}(\Omega - \bar{\omega})\bar{\omega}'\dot{\eta} + e^{\lambda}\left[-\frac{4}{r^3}e^{\lambda}m_2 + \frac{2}{r}\left(v_2' - h_2'\right) + \frac{r}{3}e^{-\nu}(\Omega - \bar{\omega})\bar{\omega}'\right]\dot{\mu}
\end{aligned}
$$

$$
\frac{R_{tA}}{\epsilon^2\zeta\Psi^{2\,*}_{0\ A}} = -e^{-\lambda}\left[\frac{1}{2}Q_2' + \frac{(\nu'-\lambda')}{4}Q_2\right] + \frac{1}{2}\dot{K}_2 + \frac{1}{2}\dot{H}_2 + \frac{1}{r}e^{\lambda}\left(h_2 + \frac{1}{r}e^{\lambda}m_2\right)\dot{\mu}
$$

$$
\begin{aligned}
\frac{R_{rA}}{\epsilon^2\zeta\Psi^{2\,*}_{0\ A}} = {} & \frac{1}{2}e^{-\nu}\dot{Q}_2 - \left(\frac{1}{2r} + \frac{\nu'}{4}\right)H_2 - \frac{1}{2}N_2' + \left(\frac{1}{2r} - \frac{\nu'}{4}\right)N_2 + \frac{1}{2}K_2' + \frac{2}{3}e^{-\nu}r^2(\Omega - \bar{\omega})j_1' \\
& + \frac{2}{3}e^{-\nu}\left[-r^2\bar{\omega}' + \left(r - r^2\frac{\nu'}{2}\right)(\Omega - \bar{\omega})\right]j_1 + \frac{1}{2}\left(\frac{1}{r}e^{\lambda}m_2 - h_2\right)\eta' - e^{2\lambda}\left(\frac{2}{r^3} + \frac{\nu'}{r^2}\right)m_2\mu
\end{aligned}
$$

$$
\begin{aligned}
\frac{R_{AB}}{\epsilon^2\zeta\Psi^{2\,*}_{0\ AB}} = {} & \frac{1}{2}(H_2 - N_2) - \frac{1}{3}e^{-\nu-\lambda}r^4\bar{\omega}'j_1' + \frac{2}{3}e^{-\nu}r^2(\Omega - \bar{\omega})j_1 - \frac{1}{6}e^{-\nu-\lambda}r^4\bar{\omega}'^2\eta \\
& + \left(-\frac{1}{3}e^{-\nu}r^3\bar{\omega}'^2 + \frac{2}{r^2}e^{2\lambda}m_2\right)\mu
\end{aligned}
$$

$$
\begin{aligned}
\frac{R_{AB}}{\epsilon^2\zeta\Phi^{2\,*}_{0\ AB}} = {} & -\frac{1}{2}e^{-\nu}r^2\ddot{K}_2 + e^{-\lambda}\bigg\{\frac{1}{2}r^2K_2'' + \left[2r + r^2\frac{(\nu'-\lambda')}{4}\right]K_2' + \left[1 - 3e^{\lambda} + r\frac{(\nu'-\lambda')}{2}\right]K_2\bigg\} \\
& + e^{-\lambda}\bigg\{-\frac{r}{2}H_2' - \left[1 + r\frac{(\nu'-\lambda')}{2}\right]H_2 - \frac{r}{2}N_2'\bigg\} + re^{-\nu-\lambda}\dot{Q}_2 + \frac{2}{3}e^{-\nu-\lambda}\bigg\{\left[-2r^4\bar{\omega}' + r^3(\Omega - \bar{\omega})\right]j_1' \\
& + \left[-r^3\bar{\omega}' + (2r^2 - r^3\nu')(\Omega - \bar{\omega})\right]j_1\bigg\} + e^{-\lambda}\left[e^{\lambda}m_2 + \frac{r^2}{2}(h_2' - v_2') + r(h_2 - v_2)\right]\eta' \\
& - \frac{2}{3}e^{-\nu-\lambda}r^4\bar{\omega}'^2\eta + \left[-\frac{4}{r}e^{\lambda}m_2 + r(v_2' - h_2') + 2(v_2 - h_2)\right]\mu' + \bigg\{e^{\lambda}\left[-\frac{4}{r}m_2' - \left(\frac{4\nu'}{r} + \frac{4\lambda'}{r}\right)m_2\right] \\
& - 2r(h_2'' - v_2'') - (5 + r\nu')(h_2' - v_2') + 2v_2' - \left(\frac{2}{r} + 2\nu'\right)(h_2 - v_2) - \frac{4}{3}e^{-\nu}r^3\bar{\omega}'^2\bigg\}\mu
\end{aligned}
$$

The principal task of this paper is to make sense out of the equations of order $\epsilon\zeta$ and the $l = 0$ equations of order $\epsilon^2\zeta$. This we do in the following sections. The algebraic computations described there were all performed by computer. The authors, with much help from Barbara Zimmerman, performed the computations on IBM 7094 computers, and later checked them on IBM 360 computers at the University of California, Santa Barbara, and at Caltech.

b) *Equations of* $O(\epsilon\zeta)$

Table 2 shows that there are two functions of r and t that should be determined by the equations of $O(\epsilon\zeta)$: the metric function j_1, and the azimuthal displacement function F, which is produced by Coriolis forces.

Take the $\epsilon\zeta$, $\Phi^1{}_0{}^*{}_A$ part of the Einstein equation $R_{rA} = 8\pi(T_{rA} - \frac{1}{2}g_{rA}T)$; multiply it by $2r^{-2}e^{\nu}$; integrate it over time; and use equation (4.2a) to eliminate μ. The result is the following differential equation for j_1:

$$j_1' = -\tfrac{1}{2}\bar{\omega}'\eta + 8\pi r^{-2}e^{\lambda+\nu/2}(E+P)(\tfrac{1}{2}\bar{\omega}'r + 2\bar{\omega})U\,. \tag{4.3a}$$

To guarantee that the metric (3.7) approaches the Minkowski form far from the star, we impose the boundary condition $j_1 = 0$ at $r = \infty$—which, together with equations (4.2c) and (4.3a), guarantees

$$j_1 = 0 \quad \text{outside the star}\,. \tag{4.3b}$$

(This condition is closely related to the absence of dipole gravitational waves; notice that j_1 is the only time-dependent metric function that appears in an $l = 1$ part of the metric.)

Take the $\epsilon\zeta$, $\Phi^1{}_0{}^*{}_A$ part of the equation $T_A{}^{\alpha}{}_{;\alpha} = 0$ (where $A = \theta, \varphi$); multiply it by $r^{-2}(E+P)^{-1}e^{\nu}$; integrate it with respect to time; and use equation (4.2a) to eliminate μ. The result is an equation for the Coriolis-induced azimuthal velocity factor $F_{,t}$:

$$F_{,t} = -j_1 - \tfrac{1}{2}\bar{\omega}\eta - \frac{\Gamma P}{E+P}\frac{e^{\nu/2}}{r^2}\bar{\omega}U' + \frac{e^{\nu/2}}{r^2}\left[\bar{\omega}' + \left(\frac{2}{r} - \frac{\nu'}{2}\right)\bar{\omega}\right]U\,. \tag{4.3c}$$

c) *Equations of* $O(\epsilon^2\zeta)$ *with* $l = 0$

Take the $\epsilon^2\zeta$, $l = 0$ part of the equation $R_{tr} = 8\pi(T_{tr} - \frac{1}{2}Tg_{tr})$; multiply it by r; integrate it with respect to time; use equation (4.3a) to eliminate j_1'; and use equation (4.2a) to eliminate μ. The result is the following initial-value equation for the metric function $H_0(t, r)$ in terms of the $\epsilon^2\zeta$, $l = 0$, radial fluid displacement W_0, and in terms of functions characterizing the nonpulsating and/or nonrotating states:

$$H_0 = 8\pi r^{-1}e^{\lambda/2}(E+P)W_0 + 8\pi\frac{e^{\lambda+\nu/2}}{r}(E+P)\left[4e^{\lambda}\frac{m_0}{r} + \left(1 + \frac{E+P}{\gamma P}\right)p_0{}^* + \tfrac{2}{3}e^{-\nu}r^2\bar{\omega}^2\right]U\,. \tag{4.4a}$$

Take the $\epsilon^2\zeta$, $l = 0$ part of the equation

$$e^{-\nu+\lambda}[R_{tt} - 8\pi(T_{tt} - \tfrac{1}{2}g_{tt}T)] + R_{rr} - 8\pi(T_{rr} - \tfrac{1}{2}g_{rr}T)\,;$$

use equations (4.2) to eliminate η' and μ, (4.3a) to eliminate j_1', and (4.4a) to eliminate H_0; use the equations of structure for a rotating star (Paper II) to simplify terms involving derivatives of $\bar{\omega}$, m_0, h_0, and $p_0{}^*$; and use the equations of structure (4.1) for a nonrotating star to simplify terms involving derivatives of ν and λ. The result is the following initial-value equation for N_0:

$$
\begin{aligned}
N_0' = &- 8\pi e^{\lambda/2} r^{-1}\Gamma P W_0' - 4\pi e^{3\lambda/2} r^{-2}(E+P)[1 + e^{-\lambda} - 8\pi r^2(\Gamma - 1)P]W_0 \\
&+ \tfrac{2}{3} e^{-\nu} r^2(\Omega - \bar{\omega})\bar{\omega}'\eta + \tfrac{4}{3} e^{\lambda-\nu} r[e^{-\lambda} r\bar{\omega}' + (1 - 3e^{-\lambda} + 8\pi r^2 P)(\Omega - \bar{\omega})]j_1 \\
&- 8\pi e^{\lambda-\nu/2} r\left[\frac{2}{3}\frac{\Gamma^2 P^2}{E+P}\bar{\omega}^2 + 2\frac{e^{\lambda+\nu}}{r^3}\Gamma P m_0 + \frac{e^{\nu}}{r^2}\Gamma(E+P)p_0^*\right]U' \\
&- \pi e^{\lambda-\nu/2}\Big\{\tfrac{2}{3} r^2(\Gamma P - E - P)\bar{\omega}'^2 + \tfrac{16}{3} r(E+P)(\Omega - \bar{\omega})\bar{\omega}' + \tfrac{64}{3}(E+P)\Omega\bar{\omega} \\
&\quad + [\tfrac{8}{3}(e^{\lambda} - 5)(E+P) - \tfrac{16}{3}\Gamma P + \tfrac{64}{3}\pi e^{\lambda} r^2 P(E+P)]\bar{\omega}^2 \\
&\quad + 8\frac{e^{2\lambda+\nu}}{r^3}(E+P)\left[2 + e^{-\lambda} - \frac{\Gamma P}{E+P} + 8\pi r^2 P\left(2 - \frac{\Gamma P}{E+P}\right)\right]m_0 \\
&\quad + \frac{e^{\lambda+\nu}}{r^2}(E+P)\left[4(e^{-\lambda} + 1)\left(1 + \frac{E+P}{\gamma P}\right)\right. \\
&\qquad \left. + 32\pi r^2\left(P - \Gamma P + \frac{E+P}{\gamma} + E + P\right)\right]p_0^*\Big\} U\,.
\end{aligned}
\tag{4.4b}
$$

Take the $\epsilon^2\zeta$, $l = 0$ part of the equation $T_r{}^{\alpha}{}_{;\alpha} = 0$; use equations (4.4a, b) to eliminate H_0 and N_0', (4.3a, c) to eliminate j_1' and $F_{,t}$, and (4.2a, b, d) to eliminate μ, η', and U''; use the equations of structure for a rotating star (Paper II) to simplify terms involving derivatives of $\bar{\omega}$, m_0, h_0, and p_0^*; and use the equations of structure (4.1) for a nonrotating star to simplify terms involving derivatives of ν and λ. Finally, assume a sinusoidal time dependence of the form

$$
\begin{aligned}
U(t, r) &= U(r) \exp{[i(\sigma^{(0)} + \epsilon^2\sigma^{(2)})t]}\ ; \\
W_0(t, r) &= W_0(r) \exp{[i(\sigma^{(0)} + \epsilon^2\sigma^{(2)})t]}\ .
\end{aligned}
\tag{4.5}
$$

Then the term $(E + P)r^{-2}e^{\lambda-\nu/2}U_{,tt}$ in the $\epsilon^0\zeta$, $l = 0$ part of $T_r{}^{\alpha}{}_{;\alpha} = 0$ acquires a piece of order $\epsilon^2\zeta$. Transfer this piece into the $\epsilon^2\zeta$, $l = 0$ part of $T_r{}^{\alpha}{}_{;\alpha}$, where it rightfully belongs. The result is the following eigenequation for $W_0(r)$:

$$
e^{-(\nu+\lambda/2)}\{-2\sigma^{(0)}\sigma^{(2)}\mathfrak{W}U - \mathfrak{L}[e^{-(\lambda+\nu)/2}W_0]\} + \mathfrak{D} = 0\ . \tag{4.6a}
$$

Here $\mathfrak{L}$ is the Chandrasekhar operator (eq. [4.2d]):

$$
\begin{aligned}
\mathfrak{L}[f] = &\, e^{(\nu+3\lambda)/2}r^{-2}(E + P)(\sigma^{(0)})^2 f + [e^{(3\nu+\lambda)/2}r^{-2}\Gamma P f']' \\
&+ e^{(3\nu+\lambda)/2}\left[-\frac{4P'}{r^3} - 8\pi e^{\lambda}\frac{P(E+P)}{r^2} + \frac{P'^2}{r^2(E+P)}\right] f\ ;
\end{aligned}
\tag{4.6b}
$$

$\mathfrak{W}$ is the "weighting function" which enters into $\mathfrak{L}$:

$$
\mathfrak{W}(r) = e^{(\nu+3\lambda)/2}r^{-2}(E + P)\ ; \tag{4.6c}
$$

and $\mathfrak{D}$ is the "driving term," which is given in table 4.

d) Expression for the Change in Frequency

Equation (4.6a) has the same structure as equation (2.7). Following the general argument of § II, we can now construct an explicit expression for the rotation-induced change in squared frequency of a radial mode. Take equation (4.6a), multiply it by

TABLE 4

THE DRIVING TERM $\mathfrak{D}$ FOR EQUATION (4.6a)

$$
\begin{aligned}
\mathfrak{D}(r) = U'\Big[\!\!\Big[& m_0 r^{-4} e^{2\lambda+\nu/2}\Gamma(E+P) + \frac{1}{2} p_0^{*} \Gamma(E+P) r^{-3} e^{\lambda+\nu/2}\Big[\frac{E+P}{\gamma P} - \frac{E}{P}\Big](1-e^{-\lambda}) \\
& - \frac{2}{3}\bar{\omega}\bar{\omega}' e^{-\nu/2}\Big[\Gamma(E+P) + 2\frac{(\Gamma P)^2}{E+P}\Big] - \frac{1}{12}\bar{\omega}'^2 r e^{-\nu/2}\Gamma(E+P) \\
& + \frac{2}{3}\bar{\omega}^2 r^{-1} e^{\lambda-\nu/2}\Gamma P\Big\{-\frac{1}{2}(3e^{-\lambda}-1)\frac{E+P}{P} + \frac{1}{2}\frac{\Gamma P}{E+P}(1-5e^{-\lambda}) + \frac{1}{2}\Gamma(1-e^{-\lambda})\Big(1-\frac{1}{\gamma}\Big) \\
& \quad + 4\pi r^2 \Gamma P\Big(1-\frac{1}{\gamma}+\frac{P}{E+P}\Big) - r e^{-\lambda}\frac{\Gamma' P}{E+P}\Big\}\Big]\!\!\Big] \\
& + U\Big[\!\!\Big[m_0 r^{-5} e^{3\lambda+\nu/2}\Big\{\Gamma(E+P)\Big[-\frac{1}{2}(1-e^{-\lambda}) + 4\pi r^2 P(1+2e^{-\lambda}) + 64\pi^2 r^4 P^2\Big] \\
& \quad +(E+P+\Gamma P)\Big[-1-3e^{-\lambda}-16\pi r^2 P\Big(1+\frac{1}{2}e^{-\lambda}\Big) - 64\pi^2 r^4 P^2\Big] + r e^{-\lambda}\Gamma' P\Big[1+8\pi r^2 P\Big] \\
& \quad - 2(E+P)[\sigma^{(0)}]^2 r^2 e^{-\lambda-\nu}\Big\} + 2h_0 r^{-2}(E+P) e^{\lambda-\nu/2}[\sigma^{(0)}]^2 \\
& + p_0^{*}(E+P) r^{-4} e^{2\lambda+\nu/2}\Big\{\Big[\frac{E+P}{\gamma P}-\frac{E}{P}\Big]\Big[-[\sigma^{(0)}]^2 r^2 e^{-\lambda-\nu} - \frac{1}{4}(1-e^{-\lambda})(1+7e^{-\lambda})\Big] + 4\pi\Gamma' P r^3 e^{-\lambda} \\
& \quad - 2\pi P r^2\Big[(1+e^{-\lambda})(2+\Gamma) + 8\pi r^2 P(1+\Gamma)\Big] \\
& \quad - 2\pi(E+P) r^2\Big[(1-e^{-\lambda})\Gamma + (1+e^{-\lambda})(2-\Gamma)\frac{1}{\gamma} + 8\pi P r^2(1-\Gamma)\Big(1+\frac{1}{\gamma}\Big)\Big]\Big\} \\
& + 4\bar{\omega}\bar{\omega}' r^{-1} e^{-\nu/2}\Big[E+P+\frac{1}{3}\Gamma P\Big] \\
& + \frac{2}{3}\bar{\omega}'^2 e^{\lambda-\nu/2}\Big\{\pi r^2\Big(1-\frac{1}{2}\Gamma\Big) P(E+P) + \pi r^2\Gamma P^2 + \frac{1}{16}\Gamma(E+P)(1-e^{-\lambda}) \\
& \quad + \frac{1}{8}(E+P+\Gamma P)(1+7e^{-\lambda}) - \frac{1}{8} r\Gamma' P e^{-\lambda}\Big\} \\
& + \frac{2}{3}\bar{\omega}^2 r^{-2} e^{\lambda-\nu/2}\Big\{-(E+P-\Gamma P)[\sigma^{(0)}]^2 r^2 e^{-\nu} + (E+P)\Big[\frac{31}{4}e^{-\lambda} - \frac{5}{2} - \frac{1}{4}e^{\lambda} + \frac{1}{2}\Gamma(e^{-\lambda}-1)\Big] \\
& \quad + \Gamma P\Big[-\frac{11}{4}e^{-\lambda}+\frac{3}{2}+\frac{1}{4}e^{\lambda}\Big] + 4\pi r^2(E+P) P(3+e^{\lambda})\Big(\frac{1}{2}\Gamma-1\Big) + 4\pi r^2\Gamma P^2(1+e^{\lambda}) \\
& \quad + 16\pi^2 r^4 P^2 e^{\lambda}\Big[(\Gamma-1)(E+P)+\Gamma P\Big] + r\Gamma' P e^{-\lambda}\Big\}\Big]\!\!\Big]
\end{aligned}
$$

$e^{\nu+\lambda/2}U$, and integrate over r from the star's center to its surface. Integrate by parts so that the equation becomes

$$\int_0^R dr[2\sigma^{(0)}\sigma^{(2)}\mathcal{W}U^2 - e^{-(\nu+\lambda)/2}\mathcal{W}_0\mathfrak{L}[U] - e^{\nu+\lambda/2}U\mathfrak{D}] = 0\,. \tag{4.7}$$

Then use $\mathfrak{L}[U] = 0$ to obtain the desired expression for $(\sigma^2)^{(2)}$:

$$(\sigma^2)^{(2)} = 2\sigma^{(0)}\sigma^{(2)} = \frac{\int_0^R dr e^{\nu+\lambda/2}U(r)\mathfrak{D}(r)}{\int_0^R dr \mathcal{W}(r)U^2(r)}\,. \tag{4.8}$$

The "driving term" $\mathfrak{D}(r)$ is given in table 4, and $\mathcal{W}(r)$ is given in equation (4.6c).

V. NEWTONIAN LIMIT

In this section we take the Newtonian limit of our expression for the rotation-induced change in frequency (eq. [4.8]) and recover the result of Chandrasekhar and Lebovitz (1968). After reinserting factors of G and c, the Newtonian limit is defined by (cf. Paper I, § VI)

$$E = GE/c^2 + O(1/c^4)\,, \tag{5.1}$$

$$P = GP/c^4 + O(1/c^6)\,, \tag{5.2}$$

$$M = GM/c^2 + O(1/c^4)\,, \tag{5.3}$$

$$e^\nu = 1 - \int_r^\infty (2GM/c^2r^2)dr + O(1/c^4)\,, \tag{5.4}$$

$$e^\lambda = 1 + 2GM/c^2r + O(1/c^4)\,, \tag{5.5}$$

$$\bar{\omega} = \Omega/c + O(1/c^3)\,, \tag{5.6}$$

$$p_0{}^* = p_0{}^*/c^2 + O(1/c^4)\,, \tag{5.7}$$

$$m_0 = Gm_0/c^2 + O(1/c^4)\,, \tag{5.8}$$

$$U = U + O(1/c^2)\,, \tag{5.9}$$

$$\sigma = \sigma/c + O(1/c^2)\,. \tag{5.10}$$

Here, we have used the same symbols on both sides of the equation to simplify notation. The quantities E, P, M, $p_0{}^*$, etc., on the left-hand side of the equations are the fully relativistic quantities having as dimensions various powers of length, while the corresponding quantities on the right-hand side are their finite Newtonian limits ($c \to \infty$) with conventional cgs dimensions involving mass, length, and time.

Inserting these limits into our pulsation equation, we find that the leading term is of $O(1/c^4)$ and has the form

$$\begin{aligned}[\sigma^2]^{(2)}EU = {} & U[\sigma^2 p_0{}^* E^2(1 - \gamma^{-1})/P - 4GEm_0/r^3 \\ & + 4GMp_0{}^*E^2(1 - \gamma^{-1})/(r^3P) + 10E\Omega^2/3] \\ & + \Gamma U'[GEm_0/r^2 - GMp_0{}^*E^2(1 - \gamma^{-1})/(r^2P) - 2E\Omega^2 r]\,. \end{aligned} \tag{5.11}$$

In our notation the expression (eq. [80]) of Chandrasekhar and Lebovitz (1968) is

$$\begin{aligned}[\sigma^2]^{(2)}\int_0^R dr[EU^2/r^2] = {} & \int_0^R dr\{Ep_0{}^*[\Gamma U'^2 - 4r^2(U^2/r^3)']/r^2 \\ & - \sigma_0{}^2E(dE/dP)p_0{}^*U^2/r^2 + 2\Omega^2EU^2/(3r^2)\}\,. \end{aligned} \tag{5.12}$$

Integrate by parts the first and second terms on the right-hand side of the Chandrasekhar-Lebovitz equation (5.12). Use the definition of γ, the Newtonian equations for rotationally perturbed stars (e.g., Paper I, § VII), and the Newtonian equation for radial pulsations (e.g., Chandrasekhar and Lebovitz 1968) to eliminate all derivatives of $p_0{}^*$, E, P, and all second derivatives of U. The result is then identical to that obtained by multiplying our equation (5.11) by U/r^2 and integrating from the star's center to its surface.

VI. SUMMARY AND CONCLUSIONS

Equation (4.8) is the chief result of this paper. Every algebraic and differential manipulation which went into this equation was performed at least twice: once using FORMAC on an IBM 7094 computer and once using FORMAC on an IBM 360.

Equation (4.8) expresses the rotation-induced change of pulsation frequency, $\sigma^{(2)}$, as a ratio of two integrals. The integrals can be evaluated from a knowledge of only (i) the nonrotating equilibrium configuration (terms of order $\epsilon^0\zeta^0$), (ii) the radial pulsations of the nonrotating configuration (terms of order $\epsilon^0\zeta$), and (iii) the rotational corrections to the structure of the equilibrium configuration (terms of order $\epsilon^2\zeta^0$). Thus, $\sigma^{(2)}$ can be calculated without ever solving any of the equations (of order $\epsilon^2\zeta$) for the coupled rotation and vibration.

After this work was completed, the authors learned that S. Chandrasekhar and J. Friedman (1971, 1972) had embarked on and completed an analogous calculation. By a rather different route Chandrasekhar and Friedman derive an equation for $\sigma^{(2)}$ which is similar in structure, and presumably equivalent, to equation (4.8).

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